A New Generalized Yang-Fourier Transforms to Heat-Conduction in a Semi-Infinite Fractal Bar

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Abstract:
The purpose of present paper to solve 1-D fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus employing the analytical Manoj Generalized Yang-Fourier transforms method.

Key Words: fractal bar, heat-conduction equation, A New Generalized Yang-Fourier transforms, Yang-Fourier transforms, local fractional calculus.

1. Introduction

A New Generalized Yang-Fourier transforms which is obtained by authors by generalization of Yang-Fourier transforms is a technique of fractional calculus for solving mathematical, physical and engineering problems. The fractional calculus is continuously growing in last five decades [1-7]. Most of the fractional ordinary differential equations have exact analytic solutions, while others required either analytical approximations or numerical techniques to be applied, among them: fractional Fourier and Laplace transforms [8,41], heat-balance integral method [9-11], variation iteration method (VIM) [12-14], decomposition method [15,41], homotopy perturbation method [16,41] etc.

The problems in fractal media can be successfully solved by local fractional calculus theory with problems for non-differential functions [25-32]. Local fractional differential equations have been applied to model complex systems of fractal physical phenomena [30-41] local fractional Fourier series method [38], Yang-Fourier transform [39,40,41] etc.

2. Generalized Yang-Fourier transform and its properties:

Let us consider $f(x)$ is a local fractional continuous in $(-\infty, \infty)$ we denote as $f(x) \in C\alpha, \beta(-\infty, \infty)$ [32, 33, 35].

Let $f(x) \in C\alpha, \beta(-\infty, \infty)$ A New Generalized Yang-Fourier transform developed by authors is written in the form [30, 31, 39, 40, 41]:

$$F_{\alpha, \beta}\{f(x)\} = f_{\omega}^{F, \alpha, \beta}(\omega) = \int_{-\infty}^{\infty} \Gamma(1 + \alpha + \beta) x^{\alpha + \beta} f(x) (dx)^{\alpha + \beta}$$

(1)

When we put $\beta$ equal to zero, and if there is no upper and lower parameter in (1) it converts in to the Yang-Fourier Transform [41].

Then, the local fractional integration is given by [30-32, 35-37, 41]:

$$\sum_{n=-\infty}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \int_{a}^{b} f(t) (dx)^{\alpha + \beta} = \int_{a}^{b} f(t) (dx)^{\alpha + \beta} = \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j) (\Delta t_j)^{\alpha + \beta}$$

(2)

where $\Delta t_{j+1} = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_3, \ldots\}$ and $\{t_0, t_1, \ldots, t_{N-1}, N \to a, t_N = b$ is a partition of the interval $[a, b]$. If $F_{\alpha, \beta}\{f(x)\} = f_{\omega}^{F, \alpha, \beta}(\omega)$, then its inversion formula takes the form [30, 31, 39, 40, 41]
\[ f(x) = F^{-1}_{a,\beta}[f^{a,\beta}(\omega)] \]

When we put \( \beta \) equal to zero, and if there is no upper and lower parameter it converts in to the Yang Inverse Fourier transform [41].

Some properties are shown as it follows [30, 31]:

Let \( F_{a,\beta}\{f(x)\} = f^{a,\beta}_{a,\beta}(\omega) \), and \( F_{a,\beta}\{g(x)\} = f^{a,\beta}_{a,\beta}(\omega) \), and let be two constants, if \((\delta)\). Then we have:

\[ F_{a,\beta}\{cf(x) + dg(x)\} = cF_{a,\beta}\{f(x)\} + dF_{a,\beta}\{g(x)\} \]

If \( \lim_{|x| \to \infty} f(x) = 0 \), then we have:

\[ F_{a,\beta}\{f^{a,\beta}(x)\} = i^{a+\beta}\omega^{a+\beta}F_{a,\beta}\{f(x)\} \]

In eq. (5) the local fractional derivative is defined as:

\[ f^{a,\beta}(x_0) = \frac{d^{a,\beta}f(x)}{dx^{a,\beta}} \bigg|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^{a,\beta}[f(x) - f(x_0)]}{(x - x_0)^{a+\beta}} \]

Where \( \Delta^{a+\beta}[f(x) - f(x_0)] \equiv \Gamma(1 + \alpha + \beta)\Delta[f(x) - f(x_0)] \).

As a direct result, repeating this process, when:

\[ f(0) = f^{a,\beta}(0) = \ldots = f^{(k-1)\alpha,(k-1)\beta}(0) = 0 \]

\[ F_{a,\beta}\{f^{k\alpha,k\beta}(x)\} = i^{a+\beta}\omega^{a+\beta}F_{a,\beta}\{f(x)\} \]

3. Heat conduction in a fractal semi-infinite bar:

If a fractal body is subjected to a boundary perturbation, then the heat diffuses in depth modeled by a constitutive relation where the rate of fractal heat flux \( \overline{q}(x, y, z, t) \) is proportional to the local fractional gradient of the temperature [32,41], namely:

\[ \overline{q}(x, y, z, t) = -K^{2a+3\beta} \nabla^{a+\beta}T(x, y, z, t) \]

Here the pre-factor \( K^{2a+3\beta} \) is the thermal conductivity of the fractal material. Therefore, the fractal heat conduction equation without heat generation was suggested in [32] as:

\[ K^{2a+3\beta} \frac{d^2(x+\beta)T(x, y, z, t)}{dx^{2(a+\beta)}} - \rho_{a+\beta}c_{a+\beta} \frac{d^{2(\alpha+\beta)}T(x, y, z, t)}{dx^{2(\alpha+\beta)}} = 0 \]

(10)

Where \( \rho_{a+\beta} \) and \( c_{a+\beta} \) are the density and the specific heat of material, respectively.

The fractal heat-conduction equation with a volumetric heat generation \( g(x, y, z, t) \) can be described as [32,41]:

\[ K^{2a+3\beta} \nabla^{2a+3\beta}T(x, y, z, t) + g(x, y, z, t)\rho_{a+\beta}c_{a+\beta} \frac{\partial^{(\alpha+\beta)}T(x, y, z, t)}{\partial t^{(\alpha+\beta)}} = 0 \]

(11)

The 1-D fractal heat-conduction equation [32,41] reads as:

\[ K^{2a+3\beta} \frac{d^2(x+\beta)T(x, t)}{dx^{2(a+\beta)}} - \rho_{a+\beta}c_{a+\beta} \frac{d^{2(\alpha+\beta)}T(x, t)}{dx^{2(\alpha+\beta)}} = 0, \quad 0 < x < \infty, t > 0 \]

(12a)

with initial and boundary conditions are:

\[ \frac{\partial^{(\alpha+\beta)}T(0, t)}{\partial t^{(\alpha+\beta)}} = \rho_{M_{q}}^{a+\beta} \omega^{a+\beta}, T(0, t) = 0 \]

(12b)

The dimensionless forms of (12a, b) are [35, 41]:

\[ \frac{\partial^{2(\alpha+\beta)}T(x, t)}{\partial x^{2(\alpha+\beta)}} = \frac{\partial^{(\alpha+\beta)}T(x, t)}{\partial x^{(\alpha+\beta)}} = 0 \]

(13a)

\[ \frac{\partial^{(\alpha+\beta)}T(0, t)}{\partial t^{(\alpha+\beta)}} = \rho_{M_{q}}^{a+\beta} \omega^{a+\beta}, T(0, t) = 0 \]

(13b)

Based on eq. (12a), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term \( g(x, t) \) is:

\[ K^{2a+3\beta} \frac{d^2(x+\beta)T(x, t)}{dx^{2(a+\beta)}} - \rho_{a+\beta}c_{a+\beta} \frac{d^{2(\alpha+\beta)}T(x, t)}{dx^{2(\alpha+\beta)}} = g(x, t), \quad -\infty < x < \infty, t > 0 \]

(14a)

With

\[ T(x, 0) = f(x), -\infty < x < \infty, \]

(14b)
The dimensionless form of the model (14a, b) is:
\[
\frac{\partial^2 (a+b) T(x, t)}{\partial x^{2(a+b)}} + \frac{\partial (a+b) T(x, t)}{\partial t^{(a+b)}} = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{15a}
\]
\[
T(x, 0) = f(x), \quad -\infty < x < \infty, \tag{15b}
\]

4. Solutions by the Generalized Yang-Fourier transform method:

Let us consider that \( F_{a,b}(T(x, t)) = T_{w}^{F,a,b}(\omega, t) \) is the Generalized Yang-Fourier transform of \( T(x, t) \), regarded as a non-differentiable function of \( x \). Applying the Yang-Fourier transform to the first term of eq. (15a), we obtain:

\[
F_{a,b} \left( \frac{\partial^2 (a+b) T(x, t)}{\partial x^{2(a+b)}} \right) = \frac{\partial (a+b)}{\partial t^{(a+b)}} T_{w}^{F,a,b}(\omega, t) \tag{16a}
\]

On the other hand, by changing the order of the local fractional differentiation and integration in the second term of eq.(15a), we get:

\[
F_{a,b} \left( \frac{\partial (a+b)}{\partial t^{(a+b)}} T(x, t) \right) = \frac{\partial (a+b)}{\partial t^{(a+b)}} T_{w}^{F,a,b}(\omega, t) \tag{16b}
\]

For the initial value condition, the Yang-Fourier transform provides:

\[
F_{a,b} \{ T(x, 0) \} = T_{w}^{F,a,b}(\omega, 0) = F_{a,b} \{ f(x) \} = f_{w}^{F,a,b}(\omega) \tag{16c}
\]

Thus we get from eqn. (16a, b, c):

\[
\frac{\partial (a+b)}{\partial t^{(a+b)}} T_{w}^{F,a,b}(\omega, t) + \omega^{2(a+b)} T_{w}^{F,a,b}(\omega, t) = 0, \quad T_{w}^{F,a,b}(\omega, 0) = f_{w}^{F,a,b}(\omega) \tag{17}
\]

This is an initial value problem of a local fractional differential equation with \( t \) as independent variable and \( w \) as a parameter.

\[
T(\omega, t) = f_{w}^{F,a,b}(\omega) p M_{q}^{a,b} (-\omega^{2(a+b)} t^{a+b}) \tag{18a}
\]

Consequently, using inversion formula, eqn. (3), we obtain:

\[
\frac{1}{(2\pi)^{a+b}} \int_{-\infty}^{\infty} p M_{q}^{a,b} (-\omega^{2(a+b)} t^{a+b}) (d\omega)^{a+b} \tag{18b}
\]

From [30, 32] we obtained,

\[
F_{a,b} \left( p M_{q}^{a,b} (-\omega^{2(a+b)}) \right) = C(\alpha + \beta) \frac{a^{a+b}}{\Gamma(1 + \alpha + \beta)} p M_{q}^{a,b} \left( -\omega^{2(a+b)} t^{a+b} \right) \tag{19a}
\]

Let \( C^{2(a+b)}/4^{a+b} = t^{a+b} \). Then we get:

\[
F_{a,b} \left( p M_{q}^{a,b} (-\omega^{2(a+b)}) \right) = \frac{4^{a+b} t^{a+b} (a^{a+b} + \beta)}{\Gamma(1 + \alpha + \beta)^{a+b}} p M_{q}^{a,b} (-\omega^{2(a+b)} t^{a+b}) \tag{19b}
\]

Thus, \( M_{w}^{F,a,b}(\omega) \) have the inverse:

\[
\frac{1}{4^{a+b} t^{a+b} (a^{a+b} + \beta)} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p M_{q}^{a,b} \left( 4^{a+b} t^{(a+b)} \alpha \beta \right) M_{w}^{F,a,b}(\omega) (d\omega)^{a+b} \tag{19c}
\]
Hence, we get:

\[ T(x, t) = (Mf)(x) = \sum_{n=0}^{\infty} \frac{(a_n)_p \cdots (a_p)_n}{(b_n)_p \cdots (b_p)_n} \Gamma(1 + \alpha + \beta) \int_{-\infty}^{\infty} f(\xi)x^{\alpha + \beta} t^{\alpha + \beta} 4^{\alpha + \beta} \frac{\xi^{\alpha + \beta}}{\pi^{\alpha + \beta}} (d\xi)^{\alpha + \beta} \]  

(20)

**Special case**

If we take \( \beta = 0 \) and if there is no upper and lower parameter then the results of a New generalized Yang Fourier Transforms convert in Yang Fourier Transforms results [41]

**Conclusions**

The communication, presented an analytical solution of 1-D heat conduction in fractal semi-infinite bar by the A New Generalized Yang-Fourier transform of non-differentiable functions.

**References**


